MATH 2050C Lecture on 2/28/2020

[Announcement: PS 4 due today, PS 5 posted]

Last time ......

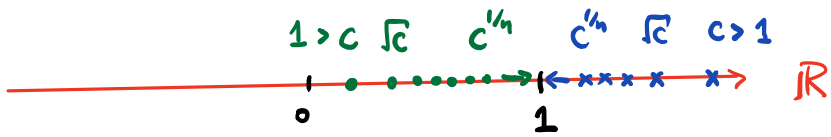
- limit if exists, then unique
- Some examples:

(1) For $b \in (0,1)$ fixed, then $\boxed{\lim(b^n) = 0}$

E.g.) $b = \frac{1}{2}$ $\quad (b^n) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots) \to 0$

Non-E.g.) $b = 2$ $\quad (b^n) = (2, 4, 8, 16, \dots) \underbrace{\text{“}\to \infty\text{”}}_{\text{not regard as convergent}}$

(2) For $c > 0$ fixed, then $\boxed{\lim(c^{1/n}) = 1}$



Note: Bernoulli's ineq: $(1+x)^n \geq 1 + nx \quad \forall x > -1, \forall n \in \mathbb{N}$ is helpful.

Sometimes, Bernoulli's is NOT sufficient.

Example: $\boxed{\lim(n^{\frac{1}{n}}) = 1}$ win

Let $\varepsilon > 0$.

Consider $n^{\frac{1}{n}} = 1 + k_n$ where $k_n \geq 0$.

By Binomial formula, $\forall n \in \mathbb{N}$,

$$n = (1+k_n)^n$$

$$= \underbrace{1 + n k_n}_{\text{Bernoulli}} + \underbrace{\tfrac{1}{2} n(n-1) k_n^2 + \cdots + k_n^n}_{\geq 0}$$

$$\geq 1 + \tfrac{1}{2} n(n-1) k_n^2$$

$|n^{\frac{1}{n}} - 1| = |k_n| = k_n$

Want: $k_n \leq$ small.

Trial 1:

$n = (1+k_n)^n$

not enough — $\geq 1 + n k_n$ Bernoulli.

$\Rightarrow \quad k_n \leq \frac{n-1}{n} \to 1$

not small

So, $k_n^2 \le \frac{2(n-1)}{n(n-1)} = \frac{2}{n} \to 0$, i.e. $k_n \le \sqrt{\frac{2}{n}}$ $(< \varepsilon)$

Choose $K = K(\varepsilon) \in \mathbb{N}$, s.t. $K > \frac{2}{\varepsilon^2}$.

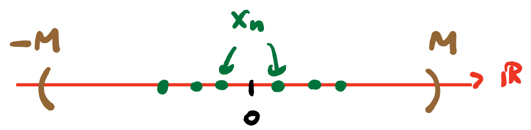
Then, $\forall n \ge K$,

$$\left| n^{\frac{1}{n}} - 1 \right| = k_n \le \sqrt{\frac{2}{n}} \le \sqrt{\frac{2}{K}} < \varepsilon .$$

$\square$

Limit Theorems (§ 3.2)

GOAL: Find some "efficient" ways to say when (x_n) is convergent, and compute its limit (if exists).



Def$^{\underline{n}}$: A seq. (x_n) is bdd

iff $\exists\, M > 0$ s.t. $|x_n| \le M \quad \forall n \in \mathbb{N}$.

(i.e. the subset $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bdd)

E.g.) $(x_n) = ((-1)^n)$ is bdd ... (*)

$(x_n) = (n)$ is unbdd.

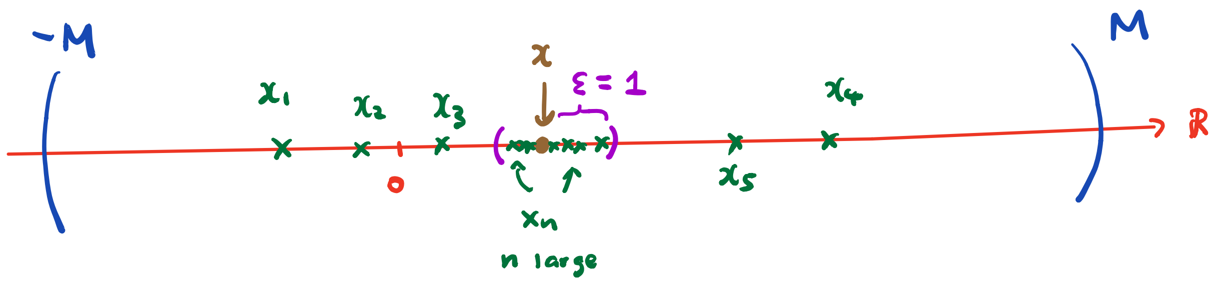
Thm: (x_n) convergent $\Rightarrow$ (x_n) bdd

Caution: "$\Leftarrow$" is false, see (*)

Cor: (x_n) unbdd $\Rightarrow$ (x_n) divergent. [Useful to show divergence]

Proof: Let (x_n) be a convergent seq., say $\lim (x_n) = x \in \mathbb{R}$.

Picture:



Let $\varepsilon = 1 > 0$, then by def$^{\underline{n}}$ of limit, $\exists\, K = K(1) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon = 1 \qquad \forall n \ge K.$$

By Triangle ineq., $\forall n \geq K$.

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| \leq \underbrace{1 + |x|}_{\text{fixed number}}$$

Then, if we take

$$M := \max\{|x_1|, |x_2|, \dots, |x_{K-1}|, 1+|x|\} > 0.$$

then clearly, $|x_n| \leq M \quad \forall n \in \mathbb{N}$

$\square$

E.g.) Fix $b > 1$, then (b^n) is unbdd $\Rightarrow$ (b^n) divergent.

(Ex: Prove this!)

Recall: $\mathbb{R}$ is a complete (sup/inf) ordered ($\leq$) field ($+,-,\times,\div$).

C? B A

Q: How is the limit process compatible with these?

Limit Thm A: Suppose (x_n), (y_n) s.t. $\lim(x_n) = x$, $\lim(y_n) = y$.

Then, (i) $\lim(x_n \pm y_n) = x \pm y$

(ii) $\lim(x_n y_n) = xy$

(iii) $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$ (*) provided that $y_n \neq 0 \ \forall n \in \mathbb{N}$ and $y \neq 0$

(Ex: Show that this is needed)

Proof: (i) Let $\varepsilon > 0$.

- Since $(x_n) \to x$, $\exists K_1 = K_1(\varepsilon/2) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon/2 \quad \forall n \geq K_1$$

- Since $(y_n) \to y$, $\exists K_2 = K_2(\varepsilon/2) \in \mathbb{N}$ s.t.

$$|y_n - y| < \varepsilon/2 \quad \forall n \geq K_2$$

$$|(x_n+y_n) - (x+y)|$$
$$= |(x_n - x) + (y_n - y)|$$
$$\leq \underbrace{|x_n - x|}_{\text{small } < \varepsilon/2} + \underbrace{|y_n - y|}_{\text{small } < \varepsilon/2} < \varepsilon$$

Take $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall n \geq K$,

$$|(x_n + y_n) - (x+y)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

(i) $\square$

(ii) Let $\varepsilon > 0$.

- Since (y_n) convergent, by Thm above, it's bdd.

i.e. $\exists\, M > 0$ s.t. $|y_n| \le M \quad \underbrace{\forall n \in \mathbb{N}}_{\text{all terms}}$

Take $M' := \max\{M, |x|\}$.

- Since $(x_n) \to x$, $\exists\, K_1 = K_1(\varepsilon/2M') \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon/2M' \quad \forall n \ge K_1$$

- Since $(y_n) \to y$, $\exists\, K_2 = K_2(\varepsilon/2M') \in \mathbb{N}$ s.t.

$$|y_n - y| < \varepsilon/2M' \quad \forall n \ge K_2$$

$$
\begin{aligned}
&|x_n y_n - xy| \\
&= |x_n y_n - x y_n + x y_n - xy| \\
&\le |y_n(x_n - x)| + |x(y_n - y)| \\
&= \underbrace{|y_n|}_{\le M'}\, \underbrace{|x_n - x|}_{\text{small } \varepsilon/2M'} + \underbrace{|x|}_{\le M'}\, \underbrace{|y_n - y|}_{\text{small } \varepsilon/2M'}
\end{aligned}
$$

For any $n \ge \bar{K} := \max\{K_1, K_2\}$, we have

$$
\begin{aligned}
|x_n y_n - xy| &= |(x_n y_n - x y_n) + (x y_n - xy)| \\
&\le |y_n|\,|x_n - x| + |x|\,|y_n - y| \\
&\le M'\,|x_n - x| + M'\,|y_n - y| \\
&< M' \cdot \frac{\varepsilon}{2M'} + M' \cdot \frac{\varepsilon}{2M'} = \varepsilon
\end{aligned}
$$

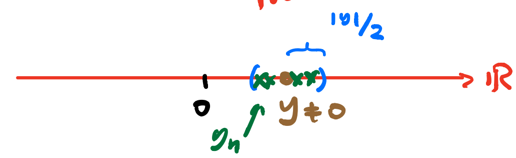
(ii) $\square$

(iii) By (ii), since $\left(\frac{x_n}{y_n}\right) = \left(x_n \cdot \frac{1}{y_n}\right)$, we just have to show

$$\lim\left(\frac{1}{y_n}\right) = \frac{1}{y} \quad \text{provided that } (*) \text{ holds}$$

Let $\varepsilon > 0$.

$$
\begin{aligned}
&\left|\frac{1}{y_n} - \frac{1}{y}\right| \\
&= \left|\frac{y_n - y}{y_n y}\right| = \frac{\overbrace{|y_n - y|}^{\text{small}}}{\underbrace{|y_n|}_{\text{Want to stay away from } 0}\,\underbrace{|y|}_{\text{fixed } \neq 0}}
\end{aligned}
$$



Claim: $\exists\, \tilde{K} \in \mathbb{N}$ s.t. $|y_n| > \frac{|y|}{2} > 0 \quad \forall n \ge \tilde{K}$.

Pf: Since $(y_n) \to y$, take $\varepsilon' := \frac{|y|}{2} \overset{(*)}{>} 0$

then $\exists\, \tilde{K} = \tilde{K}(\varepsilon') \in \mathbb{N}$ s.t.

$$|y_n - y| < \varepsilon' = \frac{|y|}{2}, \quad \forall n \ge \tilde{K}$$

i.e. $\underbrace{y - \frac{|y|}{2}}_{\frac{|y|}{2} \text{ when } y > 0.} < y_n < \underbrace{y + \frac{|y|}{2}}_{-\frac{|y|}{2} \text{ when } y < 0}, \quad \forall n \ge \tilde{K}$

Since $(y_n) \to y$, $\exists\, K' = K'\left(\varepsilon \cdot \frac{|y|^2}{2}\right) \in \mathbb{N}$ s.t.

$$|y_n - y| < \varepsilon \cdot \frac{|y|^2}{2} \qquad \forall\, n \geq K'$$

Take $K \in \mathbb{N}$ s.t. $K > \max\{\tilde{K}, K'\}$,

For any $n \geq K$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} = \frac{1}{|y_n|}\frac{1}{|y|}|y_n - y|$$

$$< \frac{1}{|y|/2} \cdot \frac{1}{|y|}|y_n - y| = \frac{2}{|y|^2}|y_n - y| < \frac{2}{|y|^2} \cdot \varepsilon \cdot \frac{|y|^2}{2} = \varepsilon$$

$\square$

Limit Thm B: Let $(x_n), (y_n)$ be convergent seq. s.t.

$$x_n \leq y_n \qquad \forall\, n \in \mathbb{N}$$

Then, $\lim(x_n) \leq \lim(y_n)$.

exist

Remark: If we assume $x_n < y_n \quad \forall\, n \in \mathbb{N}$, (Ex: find an example)

then we still just have $\lim(x_n) \leq \lim(y_n)$.

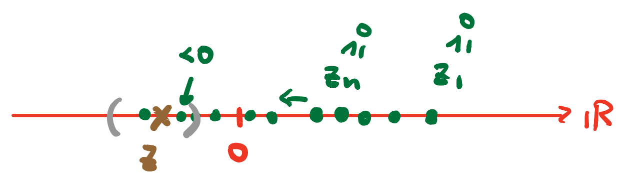
Proof: Let $(z_n) := (y_n - x_n)$. Then,

(i) $z_n \geq 0 \quad \forall\, n \in \mathbb{N}$ — by assumption

(ii) $\lim(z_n) = \lim(y_n) - \lim(x_n)$ — by Thm A

Suffices to show: $z := \lim(z_n) \geq 0$.

By Contradiction. Assume $z < 0$.



Take $\varepsilon := \frac{|z|}{2} > 0$, then $\exists\, K$ s.t.

$$|z_n - z| < \varepsilon = \frac{|z|}{2} \qquad \forall\, n \geq K$$

$$\Rightarrow \qquad z_n < z + \frac{|z|}{2} = -\frac{|z|}{2} < 0 \qquad \forall\, n \geq K$$

Contradicting that $z_n \geq 0 \quad \forall\, n \in \mathbb{N}$.

$\square$